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RITZ-GALERKIN METHODS FOR SINGULAR BOUNDARY VALUE PROBLEMS.(U)
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RITZ-GALERKIN METHODS FOR SINGULAR BOUNDARY VALUE PROBLEMS

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ABSTRACT

This paper is concerned with the application of the Ritz-Galerkin method to the numerical solution of singular boundary value problems of the type arising when Poisson's equation on a domain with cylindrical or spherical symmetry is reduced to a one-dimensional problem. The objective is to derive a priori L_2 and L_∞-norm estimates for the error. The difficulty is that these norms are not natural norms for the reduced problem. With the aid of B-splines we prove some nonstandard approximation - theoretic results and use these to derive the desired error estimates. Some numerical results are presented.

AMS (MOS) Subject Classifications: 65L10, 65N15, 65N30, 41A25

Key Words: Singular problems, Weighted spline projections, Rayleigh-Ritz-Galerkin methods

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EXPLANATION

The Ritz-Galerkin (finite element) method is a widely-studied method for obtaining numerical solutions to differential equations. Much is known about the convergence behavior of the approximate solution when all the coefficients in the differential equation are regular. This report investigates the application of the method to boundary value problems in one space dimension with a singular coefficient of the type which arises when symmetry permits the reduction of a partial differential equation to an ordinary differential equation. The basic result is that the singular coefficient does not degrade the performance of the method. Some numerical examples are given.

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RITZ-GALERKIN METHODS FOR SINGULAR BOUNDARY VALUE PROBLEMS

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§1. Introduction

When Poisson's equation $-\Delta u = f$ is encountered on a domain with cylindrical or spherical symmetry, one can, if the data depend only on the radial coordinate, make a change of variables to reduce the problem to a one-dimensional problem, albeit at the expense of introducing a singularity in the equation. This report investigates the Ritz-Galerkin process for numerically solving the one-dimensional reduced problem, with the objective of deriving optimal error bounds of the type normally encountered in finite element analysis.

In pursuing this work, the motivating ideas are roughly the following. Assuming smooth data, the solution u to Poisson's equation is smooth, hence is well-approximable by piecewise polynomials. One would thus expect a Ritz-Galerkin approximation to u from a piecewise polynomial space to be "optimally" close to u. Unfortunately, the singularity in the one-dimensional problem frustrates the usual analysis unless one either adds to the finite-dimensional space functions which match the behavior of the Green's function or makes estimates in a natural weighted Sobolev norm ([2], [13]). There have been some recent works that have made progress on this question. Dupont and Wahlbin [10] analyzed the problem $-(a^2u')' + qu = f$ on $(0,1)$ where $q(x) \geq q_0 > 0$ for $0 \leq x \leq 1$ and $a(x) \in C^1[0,1]$. They were able to show the Ritz-Galerkin process produces an approximate solution u_h which is optimally close to u in the $L_2(0,1)$ norm. Their proof seems to depend strongly on the conditions $a \in C^1[0,1]$ and $q(x)$ strictly positive. In another recent work, de Hoog and Weiss [7] have analyzed the application of collocation methods to singular boundary value problems such as the ones considered here. In a difficult piece of analysis they show that collocation gives, roughly speaking, the same results for singular boundary value problems as for nonsingular

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problems (for which see [6] or [14]). Their results gave further impetus to the hope that the Ritz-Galerkin method would give the same results for singular problems as for nonsingular problems.

This paper divides into three parts. In section 2 we prove some results of a purely approximation-theoretic nature concerning the convergence of some weighted projection operators. In section 3 these results are applied to give a priori error estimates in L_2 and L_∞ for the basic problem. Some of these results are extended to nonlinear problems in section 4, and some numerical examples are presented.

I would like to thank Carl de Boor for several very helpful conversations in the course of this work.

§2. Convergence of Weighted Projections

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of the interval $I = (0,1)$. Define $I_i := (x_{i-1}, x_i)$, $h_i := x_i - x_{i-1}$, $h := \max_i h_i$. For J an interval, let $\Pi_k(J)$ denote the set of polynomials of order k (degree $< k$) on J. Let $0 \le \nu \le k-1$ be integers. Let $S^h = S^h_{k,\nu} := \{v \in H^\nu(I) : v|_{I_i} \in \Pi_k(I_i),\ 1 \le i \le N\}$, where $H^\nu(I)$ denotes the usual Sobolev space of functions with ν weak derivatives on I, equivalently, the space of functions in $C^{\nu-1}(I)$ with $v^{(\nu-1)}$ absolutely continuous and $v^{(\nu)} \in L_2(I)$.

Define the sequence $\underline{\underline{t}} = (t_i)_1^{n+k} = (x_0,\ldots,x_0,x_1,\ldots,x_1,\ldots,x_N,\ldots,x_N)$ where x_0, x_N are repeated k times and x_i is repeated $k - \nu$ times for $1 \le i \le N-1$. It is well-known that S^h has dimension $n = N(k-\nu) + \nu$, and a convenient basis consists of the normalized B-splines, $\{B_i\}_1^n$. We have $\operatorname{supp} B_i \subseteq [t_i, t_{i+k}]$, $B_i(x) \ge 0$, and $\Sigma_1^n B_i(x) = 1$ for all $x \in I$. For these and other facts about B-splines, see, e.g. [5].

Let $\beta \ge 0$. Define a map $P_\beta = P_{\beta,h} : L_2(I) \to S^h$ via $\int x^\beta (u - P_\beta u)\varphi\, dx = 0$ for all $\varphi \in S^h$. For $\beta = 0$, $P_\beta u$ is the L_2-projection of u onto S^h, and estimates of the error $\|u - P_\beta u\|_{L_p}$ are well-known for $p = 2$ and $p = \infty$. Our objective is to derive a priori estimates for $\|u - P_\beta u\|_{L_p}$ for $\beta > 0$ and $p = 2,\infty$.

We will make the following assumption on the mesh: there are constants $M > 0$, $\gamma \ge 1$ independent of h such that

$$x_j/x_i \le M(j/i)^\gamma \quad \text{for } 1 \le i \le j . \tag{2.1}$$

Define the global mesh ratio by $M_{\underline{\underline{t}}} := \max_{i,j} (t_{i+k} - t_i)/(t_{j+k} - t_j)$. We can now state the main result of this section.

Theorem 2.1. (1) P_β is bounded as a map on L_2,

$$\|P_\beta\|_2 \le C(k,\beta,M,\gamma) .$$

(2) P_β is bounded as a map on L_∞,

$$\|P_\beta\|_\infty \le M_{\underline{\underline{t}}}^{1/2} \cdot C(k,\beta,M,\gamma) .$$

As an immediate corollary, we have

Corollary 2.2. (1) $\|P_\beta u - u\|_{L_2} \le C(k,\beta,M,\gamma) \inf_{u_h} \|u - u_h\|_{L_2}$.

(2) $\|P_\beta u - u\|_{L_\infty} \le M_{\underline{\underline{t}}}^{1/2} \cdot C(k,\beta,M,\gamma) \cdot \inf_{u_h} \|u - u_h\|_{L_\infty}$ if $u \in L_\infty(I)$.

The proof of the Corollary just uses the fact that P_β is a projector, in addition to Theorem 2.1.

The proof of Theorem 2.1 will follow as a sequence of lemmas. The basic approach is to write down the matrix representing the map P_β with respect to the B-spline basis and then make some careful estimates on the matrix problem.

Lemma 2.3. The assumption (2.1) implies

$$t_{i+k} \le C(M,\gamma,k)t_i \quad \text{for} \quad k+1 \le i \le n\,, \tag{2.2}$$

$$t_{j+k}/t_{i+k} \le C(M,\gamma,k)(j/i)^\gamma \quad \text{for} \quad 1 \le i \le j \le n\,. \tag{2.3}$$

Proof. We have $x_i = t_{i(k-\nu)+j}$ for $1 \le i \le N-1$, $1+\nu \le j \le k$, $x_0 = t_j$ for $1 \le j \le k$, and $x_N = t_{N(k-\nu)+j}$ for $1+\nu \le j \le k+\nu$. Hence $t_i = x_{p(i)}$ where $p(i) = N$ if $n+1 \le i \le n+k$, $p(i) = 0$ if $1 \le i \le k$, and $p(i) = \lfloor (i-\nu-1)/(k-\nu) \rfloor$ if $k+1 \le i \le n$, where $\lfloor\ \rfloor$ denotes the greatest integer function. Thus $t_{i+k}/t_i = x_{p(i+k)}/x_{p(i)} \le M(p(i+k)/p(i))^\gamma$.

For $k+1 \le i \le n-k$ we have

$$\frac{p(i+k)}{p(i)} = \frac{\lfloor (i-1)/(k-\nu) \rfloor + 1}{\lfloor (i-\nu-1)/(k-\nu) \rfloor} \le k+1\,,$$

and for $n < i+k \le n+k$ we have

$$\frac{p(i+k)}{p(i)} = \frac{N}{\lfloor (i-\nu-1)/(k-\nu) \rfloor} \le \frac{N}{\max(1,N-k)} \le k+1\,.$$

Thus $t_{i+k}/t_i \le M(k+1)^\gamma = C(M,\gamma,k)$, which proves (2.2).

Now consider (2.3). If $i = 1$, $t_{j+k}/t_{i+k} \le x_j/x_1 \le Mj^\gamma$. If $i > 1$,

$t_{j+k}/t_{i+k} = x_{p(j+k)}/x_{p(i+k)} \le M(p(j+k)/p(i+k))^\gamma < M((j-1+k-\nu)/(i-1))^\gamma \le M((1+k)j/i)^\gamma$,

which proves (2.3) with $C(M,\gamma,k) = M(1+k)^\gamma$.

Lemma 2.4. For $1 \le i \le n$,

$$\int x^{\beta} B_i(x)dx \ge C(M,\gamma,k,\beta)t_{i+k}^{\beta} \int B_i(x)dx . \tag{2.4}$$

Proof. For $k + 1 \le i \le n$, we have

$$\int x^{\beta} B_i \ge t_i^{\beta} \int B_i = (t_i/t_{i+k})^{\beta}\, t_{i+k}^{\beta} \int B_i \ge C(M,\gamma,k)^{-\beta} t_{i+k}^{\beta} \int B_i$$

by (2.2).

For $1 \le i \le k$, $\int x^{\beta} B_i = \int_0^{x_1} x^{\beta} B_i + \int_{x_1}^{t_{i+k}} x^{\beta} B_i$. On $[0,x_1]$, B_i is a polynomial of order k. By the equivalence of all norms on a finite-dimensional space,

$$\int_0^{x_1} x^{\beta} B_i \ge C(k,\beta) x_1^{\beta} \int_0^{x_1} B_i .$$

Hence

$$\begin{aligned}
\int x^{\beta} B_i &\ge C(k,\beta) x_1^{\beta} \int_0^{x_1} B_i + x_1^{\beta} \int_{x_1}^{t_{i+k}} B_i \\
&\ge \min(1,C(k,\beta)) \cdot x_1^{\beta} \int B_i \\
&= C'(k,\beta)(t_{k+1}/t_{i+k})^{\beta} t_{i+k}^{\beta} \int B_i \\
&\ge C'(k,\beta)\; C(M,\gamma,k)^{-\beta} i^{-\beta\gamma} t_{i+k}^{\beta} \int B_i \\
&\ge C'(k,\beta)\; C(M,\gamma,k)^{-\beta} k^{-\gamma\beta} t_{i+k}^{\beta} \int B_i \\
&= C(M,\gamma,k,\beta) t_{i+k}^{\beta} \int B_i .
\end{aligned}$$

Lemma 2.5. For $1 \le i \le n$, there exists $f_{i,\beta} \in L_\infty(I)$ with $\operatorname{supp} f_{i,\beta} \subseteq [t_i,t_{i+k}]$, $\|f_{i,\beta}\|_{L_\infty} \le C(M,\gamma,k,\beta)(t_{i+k} - t_i)^{-1} t_{i+k}^{-\beta}$, and $\int x^{\beta} f_{i,\beta}(x) B_j(x)dx = \delta_{ij}$ for $1 \le j \le n$.

Proof. By [4], we must construct $f_{i,\beta}$ so that $x^{\beta} f_{i,\beta}(x) = f^{(k)}(x)$ where $f^{(k)} \in L_\infty(t_i,t_{i+k})$ with

$$f = \begin{cases} 0 & k\text{-fold at } t_i \\ 0 & \text{at all } t_j \in (t_i,t_{i+k}) \\ \psi_i & k\text{-fold at } t_{i+k} \end{cases} \qquad (\psi_i(t) := (t - t_{i+1}) \cdots (t - t_{i+k-1})/(k-1)!) . \tag{2.5}$$

If $t_i > 0$ we can simply use the construction in [4] and get the bound

$$\|f_{i,\beta}\|_{L_\infty} = \|x^{-\beta} f^{(k)}\|_{L_\infty}$$

$$\leq t_i^{-\beta} D_k (t_{i+k} - t_i)^{-1}$$

$$= (t_{i+k}/t_i)^{\beta} D_k t_{i+k}^{-\beta} (t_{i+k} - t_i)^{-1}$$

$$\leq C(M,\gamma,k,\beta) D_k t_{i+k}^{-\beta} (t_{i+k} - t_i)^{-1} .$$

If $t_i = 0$ a slightly different construction suffices. Let $\delta := t_{i+k}/4$, and let $G^{\#}$ be defined by

$$G^{\#}(x) = \begin{cases} 0 & \text{for } x \leq \delta \\ 1 & \text{for } x \geq 3\delta \\ (x - \delta)/2\delta & \text{for } \delta < x < 3\delta \end{cases} .$$

Let $\varphi \in C^{\infty}(\mathbb{R})$ with $\operatorname{supp} \varphi \subseteq [-1,1]$, $\varphi \geq 0$, $\int \varphi = 1$. For any $\varepsilon > 0$, let $\varphi_\varepsilon(x) := \varphi(x/\varepsilon)$.

Now let $\varepsilon = \delta/2$ and let $G = G^{\#} * \varphi_\varepsilon$, i.e., $G(x) = \int G^{\#}(y)\varphi_\varepsilon(x - y)dy$. Then $G \in C^{\infty}(\mathbb{R})$ with $G(x) = 0$ for $x \leq \varepsilon$, $G(x) = 1$ for $x \geq \frac{7}{8} t_{i+k}$.

We have $|G^{(j)}(x)| \leq \varepsilon^{-j} \|G^{\#}\|_{L_\infty} \|\varphi^{(j)}\|_{L_1} \leq \varepsilon^{-j} \|\varphi^{(j)}\|_{L_1} =: \varepsilon^{-j} M_j$. Let $f(x) := G(x)\psi_i(x)$. Then f satisfies (2.5). Let $f_{i,\beta}(x) := x^{-\beta} f^{(k)}(x)$. To estimate $\|f_{i,\beta}\|_{L_\infty}$, we note that $f_{i,\beta}(x) = 0$ for $x < \varepsilon$, hence

$$|f_{i,\beta}(x)| \leq \varepsilon^{-\beta} \sum_{j=0}^{k} \binom{k}{j} |\psi_i^{(k-j)}(x)|\,|G^{(j)}(x)|$$

$$\leq 8^{\beta} t_{i+k}^{-\beta} \sum_{j=0}^{k} \binom{k}{j} t_{i+k}^{j-1} 8^{j} t_{i+k}^{-j} M_j$$

$$\leq C(k,\beta) t_{i+k}^{-\beta} (t_{i+k} - t_i)^{-1} ,$$

and the proof is complete.

For $u(x)$ given, $P_\beta u := \Sigma\, \alpha_i B_i$ is obtained by solving the linear system

(2.6) $$G_0 \underline{\underline{\alpha}} = \underline{\underline{u}} ,$$

where $(G_0)_{ij} := \int x^\beta B_i B_j dx$, $u_i := \int x^\beta u B_i dx$. Define $\tilde{B}_i := B_i/(\int x^\beta B_i)^{1/2}$, $D := \operatorname{diag}((\int x^\beta B_i)^{1/2})$, $(G_\beta)_{ij} := \int x^\beta \tilde{B}_i \tilde{B}_j dx$. Then $DG_\beta D = G_0$, so $G_0 \underline{\underline{\alpha}} = \underline{\underline{u}}$ is equivalent to $\underline{\underline{\alpha}} = (D^{-1}G_\beta^{-1}D)D^{-2}\underline{\underline{u}}$. But we have

$$\|P_\beta u\|_{L_\infty} = \|\Sigma\, \alpha_i B_i\|_{L_\infty} \leq \|\underline{\underline{\alpha}}\|_\infty \quad \text{and} \quad \|D^{-2}\underline{\underline{u}}\|_\infty \leq \|u\|_{L_\infty} .$$

Hence

$$\|P_\beta u\|_{L_\infty} \leq \|D^{-1}G_\beta^{-1}D\|_\infty \, \|u\|_{L_\infty} . \tag{2.7}$$

Lemma 2.6. There exists a constant $K_0 = K_0(M,\gamma,k,\beta)$ such that for all $\underline{\underline{a}} = (a_i)_1^n$,

$$K_0 \|\underline{\underline{a}}\|_2 \leq \|G_\beta \underline{\underline{a}}\|_2 \leq \|\underline{\underline{a}}\|_2 . \tag{2.8}$$

Proof. Given $\underline{\underline{a}}$, let $a_h := \Sigma\, a_i \tilde{B}_i$. Then, since

$$\begin{aligned} a_h(x)^2 &= (\Sigma\, a_i B_i(x)/(\int x^\beta B_i)^{1/2})^2 \\ &\leq \Sigma\, a_i^2 B_i(x)/\int x^\beta B_i , \end{aligned}$$

we have

$$\underline{\underline{a}}^T G_\beta \underline{\underline{a}} = \int x^\beta a_h(x)^2 dx \leq \Sigma\, a_i^2 = \underline{\underline{a}}^T \underline{\underline{a}} .$$

Since G_β is symmetric positive definite, this yields the upper bound in (2.7).

For the lower bound, take $\underline{\underline{a}} = (a_i)$, and again let $a_h := \Sigma\, a_i \tilde{B}_i$. By Lemma 2.5, we have

$$(\int x^\beta B_i)^{-1/2} a_i = \int_{t_i}^{t_{i+k}} x^\beta f_{i,\beta}(x) a_h(x) dx,$$

and hence

$$\begin{aligned} (\int x^\beta B_i)^{-1} a_i^2 &\leq C(M,\gamma,k,\beta)^2 (t_{i+k} - t_i)^{-2} t_{i+k}^{-2\beta} \cdot \int_{t_i}^{t_{i+k}} x^\beta dx \cdot \int_{t_i}^{t_{i+k}} x^\beta a_h(x)^2 dx \\ &\leq C(M,\gamma,k,\beta)^2 (t_{i+k} - t_i)^{-1} t_{i+k}^{-\beta} \int_{t_i}^{t_{i+k}} x^\beta a_h(x)^2 dx . \end{aligned}$$

Thus

$$a_i^2 \le C(M,\gamma,k,\beta)^2(t_{i+k} - t_i)^{-1} t_{i+k}^{-\beta} \cdot \int x^\beta B_i dx \cdot \int_{t_i}^{t_{i+k}} x^\beta a_h(x)^2 dx$$

$$\le C(M,\gamma,k,\beta)^2 \cdot k^{-1} \cdot \int_{t_i}^{t_{i+k}} x^\beta a_h(x)^2 dx ,$$

since $\int B_i = (t_{i+k} - t_i)/k$. A summation on i gives

$$\underline{\underline{a}}^T \underline{\underline{a}} \le C(M,\gamma,k,\beta)^2 \int x^\beta a_h(x)^2 dx = C(M,\gamma,k,\beta)^2 \underline{\underline{a}}^T G_\beta \underline{\underline{a}} ,$$

which gives the lower bound in (2.8).

Now, Demko [8] proved the following result. If A is a band matrix satisfying $\underline{K} \|\underline{\underline{v}}\|_q \le \|A\underline{\underline{v}}\|_q \le \overline{K} \|\underline{\underline{v}}\|_q$ for some $1 \le q \le \infty$ and $\underline{K} > 0$, then the entries of $A^{-1} = (\alpha_{ij})$ decay exponentially away from the diagonal, in the sense that there exist constants $C > 0$ and $\lambda \in (0,1)$ depending only on $\underline{K}$, $\overline{K}$ and the bandwidth of A such that $|\alpha_{ij}| \le C\lambda^{|i-j|}$. This result combined with (2.8) allows us to conclude the existence of constants $C > 0$, $\lambda \in (0,1)$, both *depending only on* M,γ,k,β, such that

$$|(G_\beta^{-1})_{ij}| \le C\lambda^{|i-j|} . \tag{2.9}$$

We can use (2.9) to estimate $\|D^{-1}G_\beta^{-1}D\|_\infty$. Indeed,

$$|(D^{-1}G_\beta^{-1}D)_{ij}| = |(\int x^\beta B_i)^{-1/2}(G_\beta^{-1})_{ij}(\int x^\beta B_j)^{1/2}|$$

$$\le C\lambda^{|i-j|}(\int x^\beta B_j)^{1/2}(\int x^\beta B_i)^{-1/2}$$

$$\le C\lambda^{|i-j|} t_{j+k}^{\beta/2}(t_{j+k} - t_j)^{1/2} \cdot C(M,\gamma,k,\beta)^{-1/2} t_{i+k}^{-\beta/2}(t_{i+k} - t_i)^{-1/2}$$

$$\le C' \cdot M_{\underline{\underline{t}}}^{1/2} \cdot (t_{j+k}/t_{i+k})^{\beta/2}\lambda^{|i-j|} .$$

Thus

$$\sum_{j=1}^{n} |(D^{-1}G_\beta^{-1}D)_{ij}| \le C' M_{\underline{\underline{t}}}^{1/2}\left(\sum_{j=1}^{i} \lambda^{|i-j|} + \sum_{j=i+1}^{n} C(M,\gamma,k)^{\beta/2}(j/i)^{\gamma\beta/2}\lambda^{|i-j|}\right)$$

$$< C' M_{\underline{\underline{t}}}^{1/2}((1-\lambda)^{-1} + C(M,\gamma,k,\beta))$$

$$\le C M_{\underline{\underline{t}}}^{1/2} ,$$

and hence

$$\|D^{-1}G_\beta^{-1}D\|_\infty = \sup_i \sum_{j=1}^{n} |(D^{-1}G_\beta^{-1}D)_{ij}| \le CM_{\underline{t}}^{1/2} ,$$

where C depends only on M,γ,k,β. With (2.7), this implies the second conclusion of Theorem 2.1.

For the L_2 bound, we write the matrix problem (2.6) slightly differently. Let $\Delta := \mathrm{diag}((\int B_i)^{1/2})$, $E := \mathrm{diag}((\int x^{2\beta}B_i^2)^{1/2})$. Then we have

$$\|P_\beta u\|_{L_2}^2 = \int (\Sigma\, \alpha_i B_i)^2 \le \Sigma\, \alpha_i^2 \int B_i = \|\Delta\underline{\underline{\alpha}}\|_2^2 ,$$

and

$$|(E^{-1}\underline{\underline{u}})_i| = |(\int x^\beta u B_i)(\int x^{2\beta}B_i^2)^{-1/2}| \le \left(\int_{t_i}^{t_{i+k}} u^2\right)^{1/2} ,$$

so

$$\|E^{-1}\underline{\underline{u}}\|_2^2 \le k\,\|u\|_{L_2}^2 .$$

Now, $G_0\underline{\underline{\alpha}} = \underline{\underline{u}}$ is equivalent to $\Delta\underline{\underline{\alpha}} = (\Delta G_0^{-1}E)E^{-1}\underline{\underline{u}}$, so

$$\|P_\beta u\|_{L_2} \le \|\Delta\underline{\underline{\alpha}}\|_2 \le \|\Delta G_0^{-1}E\|_2\, \|E^{-1}\underline{\underline{u}}\|_2 \le \|\Delta G_0^{-1}E\|_2\, k^{1/2}\,\|u\|_{L_2} . \tag{2.10}$$

But $\Delta G_0^{-1}E = \Delta D^{-1}G_\beta^{-1}D^{-1}E = (\Delta D^{-1}G_\beta^{-1}D\Delta^{-1})(\Delta D^{-2}E)$, so

$$\|\Delta G_0^{-1}E\|_2 \le \|(D\Delta^{-1})^{-1}G_\beta^{-1}D\Delta^{-1}\|_2\, \|\Delta D^{-2}E\|_2 . \tag{2.11}$$

We can estimate each of the latter two quantities. Indeed,

$$\Delta D^{-2}E = \mathrm{diag}((\int B_i)^{1/2}(\int x^\beta B_i)^{-1}(\int x^{2\beta}B_i^2)^{1/2}) ,$$

so

$$\begin{aligned} \|\Delta D^{-2}E\|_2 &\le \max_{1\le i\le n} (\int B_i)^{1/2} C(M,\gamma,k,\beta)^{-1} t_{i+k}^{-\beta} (\int B_i)^{-1} t_{i+k}^{\beta} (\int B_i)^{1/2} \\ &\le C(M,\gamma,k,\beta)^{-1} . \end{aligned} \tag{2.12}$$

Now we prove a final lemma.

Lemma 2.7. Let A be an $n \times n$ matrix whose inverse $A^{-1} = (\alpha_{ij})$ satisfies $|\alpha_{ij}| \le c\lambda^{|i-j|}$ where $c > 0$, $0 < \lambda < 1$. Let $D = \mathrm{diag}(d_i)$ satisfy $d_i > 0$ for

$1 \le i \le n$ and

$$d_j/d_i \le \begin{cases} K & \text{if } 1 \le j \le i \\ K(j/i)^m & \text{if } 1 \le i \le j \end{cases} .$$

Then $B := D^{-1}A^{-1}D$ satisfies $\|B\|_2 \le C(c,\lambda,K,m)$.

Proof. Take $\underline{x} = (x_i)_1^n$. We have $(B\underline{x})_i = \sum_{j=1}^n d_i^{-1} d_j \alpha_{ij} x_j$, so

$$\|B\underline{x}\|_2^2 = \sum_{i=1}^n \left(\sum_{j=1}^n d_i^{-1} d_j \alpha_{ij} x_j\right)^2$$

$$\le \sum_{i=1}^n \left(\sum_{j=1}^n d_i^{-2} d_j^2 |\alpha_{ij}|\right) \cdot \left(\sum_{j=1}^n |\alpha_{ij}| x_j^2\right) .$$

For each fixed i, we have

$$\sum_{j=1}^n d_i^{-2} d_j^2 |\alpha_{ij}| \le K^2 \sum_{j=1}^i |\alpha_{ij}| + K^2 i^{-2m} \sum_{j=i+1}^n j^{2m} |\alpha_{ij}|$$

$$\le K^2 c \sum_{j=1}^i \lambda^{|i-j|} + K^2 i^{-2m} c \sum_{j=i+1}^n j^{2m} \lambda^{|i-j|}$$

$$\le K^2 c (1-\lambda)^{-1} + K^2 c \cdot C(m,\lambda)$$

$$= C'(c,\lambda,K,m) .$$

Hence

$$\|B\underline{x}\|_2^2 \le C'(c,\lambda,K,m) \sum_i \sum_j |\alpha_{ij}| x_j^2$$

$$= C'(c,\lambda,K,m) \sum_{j=1}^n x_j^2 \sum_{i=1}^n |\alpha_{ij}|$$

$$\le C'(c,\lambda,K,m) \cdot c \cdot 2(1-\lambda)^{-1} \sum_{j=1}^n x_j^2$$

$$= C(c,\lambda,K,m) \|\underline{x}\|_2^2 ,$$

which completes the proof.

To apply the Lemma, we note that

$$(D\Delta^{-1})_{ii} = \left(\int B_i\right)^{-1/2} \left(\int x^\beta B_i\right)^{1/2}$$

and hence

$$(D\Delta^{-1})_{jj}/(D\Delta^{-1})_{ii} = (\int B_j)^{-1/2}(\int x^\beta B_j)^{1/2}(\int x^\beta B_i)^{-1/2}(\int B_i)^{1/2}$$

$$\leq t_{j+k}^{\beta/2}\, C(M,k,\gamma,\beta)\, t_{i+k}^{-\beta/2}$$

$$\leq \begin{cases} C(M,k,\gamma,\beta) & \text{if } 1 \leq j \leq i \\ C(M,k,\gamma,\beta)(j/i)^{\gamma\beta/2} & \text{if } 1 \leq i \leq j . \end{cases}$$

By Lemma 2.7, $\|(D\Delta^{-1})^{-1}G_\beta^{-1}D\Delta^{-1}\|_2 \leq C(M,k,\beta,\gamma)$. By (2.10), (2.11) and (2.12), this implies

$$\|P_\beta u\|_{L_2} \leq C(M,k,\beta,\gamma)\, \|u\|_{L_2} ,$$

which completes the proof of Theorem 2.1.

§3. A Ritz-Galerkin Method

Consider the model problem, where $\alpha \geq 1$,

(3.1)
$$-u''(x) - \frac{\alpha}{x} u'(x) = f(x) \quad \text{on} \quad (0,1) ,$$
$$u(1) = 0, \; u'(0) = 0 .$$

Such an equation arises, with $\alpha = n - 1$, when one changes variables in a rotationally symmetric Poisson's equation in $\mathbb{R}^n$. Let us write (3.1) in the form

(3.2)
$$-(x^\alpha u')' = x^\alpha f \quad \text{on} \quad (0,1) ,$$
$$u(1) = 0, \; u'(0) = 0 .$$

Let $0 \leq a < b$. Define $H^1_\alpha(a,b) := \{v \in L_2(a,b) : \int_a^b x^\alpha (v^2 + (v')^2)dx < \infty\}$, $\overset{\circ}{H}{}^1_\alpha(a,b) := H^1_\alpha(a,b) \cap \{v \in C(a,b] : v(b) = 0\}$. We will have need of the following Poincaré-type inequality.

<u>Lemma 3.1</u>. Let $v \in \overset{\circ}{H}{}^1_\alpha(a,b)$. Then $\|x^{\alpha/2}v\|_{L_2(a,b)} \leq 2(b - a)\, \|x^{\alpha/2}v'\|_{L_2(a,b)}$.

<u>Proof</u>. *We have, integrating by parts,*

$$\int_a^b x^\alpha v(x)^2 dx = -\frac{2}{\alpha + 1}\int_a^b (x^{\alpha+1} - a^{\alpha+1}) v(x) v'(x) dx$$

$$\leq \frac{2}{\alpha + 1} \|x^{\alpha/2}v\|_{L_2(a,b)} \|(x^{\alpha+1} - a^{\alpha+1})x^{-\alpha/2}v'\|_{L_2(a,b)}$$

$$\leq \frac{2}{\alpha + 1} \|x^{\alpha/2}v\|_{L_2(a,b)} \|(x^{\alpha+1} - a^{\alpha+1})x^{-\alpha}\|_{L_\infty(a,b)} \|x^{\alpha/2}v'\|_{L_2(a,b)}$$

$$\leq \frac{2}{\alpha + 1} \cdot (b^{\alpha+1} - a^{\alpha+1})b^{-\alpha} \|x^{\alpha/2}v\|_{L_2(a,b)} \|x^{\alpha/2}v'\|_{L_2(a,b)} .$$

An elementary computation shows $(b^{\alpha+1} - a^{\alpha+1})b^{-\alpha} \leq (\alpha + 1)(b - a)$ if $a > 0$, while if $a = 0$ the upper bound is trivial. Thus the lemma holds. A more general version of this result may be found in [2].

For $u,v \in H^1_\alpha(I)$, define $(u,v)_E := \int_0^1 x^\alpha u'v'dx$, $\|u\|_E := (u,u)_E^{1/2}$. Consider the following problem.

<u>Problem (P)</u>. Find $u \in \overset{\circ}{H}{}^1_\alpha(I)$ such that for all $v \in \overset{\circ}{H}{}^1_\alpha(I)$, $(u,v)_E = \int_0^1 x^\alpha f(x)v(x)dx$.

Let us assume that $x^{\alpha/2}f \in L_2(I)$. It is easy to show that a unique solution u to Problem (P) exists and that $\|u\|_E \leq 2\|x^{\alpha/2}f\|_{L_2}$.

We now define the finite-dimensional problem. Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be any partition of I, and let $S^h = S^h_{k,\nu}$ be defined as in §2, with $1 \leq \nu \leq k-1$. Let $\overset{\circ}{S}{}^h := \{\varphi \in S^h : \varphi(1) = 0\}$, and consider the following problem.

Problem (P_h). Find $u_h \in \overset{\circ}{S}{}^h$ such that for all $v_h \in \overset{\circ}{S}{}^h$, $(u_h, v_h)_E = \int_0^1 x^\alpha f(x) v_h(x)dx$.

It is again easy to show a unique solution u_h exists and is characterized by

$$(3.3) \qquad (u - u_h, v_h)_E = \int x^\alpha (u - u_h)' v_h' dx = 0$$

for all $v_h \in \overset{\circ}{S}{}^h$, equivalently

$$(3.4) \qquad \|u - u_h\|_E = \inf_{v_h \in \overset{\circ}{S}{}^h} \|u - v_h\|_E ,$$

where u is the solution to Problem (P). Since $\|v\|_E = \|x^{\alpha/2}v'\|_{L_2}$, (3.4) indicates the natural norms in this problem are the weighted L_2 norms. We will pursue this a little further and show how Nitsche's trick carries through. We need two lemmas.

Lemma 3.2. Assume the partition (x_i) satisfies $x_j/x_i \leq M(j/i)^\gamma$ for $1 \leq i \leq j$. Then there exists a $C = C(M,\gamma,\alpha,k)$ such that for all $z \in H^1_\alpha(I)$,

$$(3.5) \qquad \inf_{z_h \in S^h} \|x^{\alpha/2}(z - z_h)\|_{L_2(I)} \leq Ch\|x^{\alpha/2}z'\|_{L_2(I)} .$$

Proof. Given z, we construct z_h in two stages. For the first stage define a step function $\zeta(x)$ by $\zeta(x) = z(x_i)$ for $x_{i-1} < x \leq x_i$, $1 \leq i \leq N$, and $\zeta(0) = \zeta(x_1)$. This makes sense because $z \in H^1_\alpha(I)$ implies z is continuous on $(0,1]$.

Now, for $1 \leq m \leq N$ we have

$$\|x^{\alpha/2}(z - \zeta)\|^2_{L_2(I_m)} = \int_{I_m} x^\alpha (z(x) - z(x_m))^2 dx .$$

The integrand $z(x) - z(x_m)$ is a function in $H^1_\alpha(I_m)$ which vanishes at x_m. By Lemma 3.1, the integral is bounded by $4(x_m - x_{m-1})^2 \int_{I_m} x^\alpha z'(x)^2 dx$. Summing from $m = 1$ to N establishes the inequality

$$(3.6) \qquad \|x^{\alpha/2}(z - \zeta)\|_{L_2} \leq 2h\|x^{\alpha/2}z'\|_{L_2} .$$

The second stage of the construction involves finding a smoother approximation z_h to the step function ζ. Let $z_h := \Sigma_{i=1}^{n} \zeta(t_i)B_i$. Then for any $x \in I$,

$$\zeta(x) - z_h(x) = \sum_{i=1}^{n} (\zeta(x) - \zeta(t_i))B_i(x) .$$

Suppose $x \in I_1$. Then, since $B_i \equiv 0$ on I_1 for $i > k$, we have

$$\zeta(x) - z_h(x) = \sum_{i=1}^{k} (\zeta(x) - \zeta(t_i))B_i(x)$$

$$= \sum_{i=1}^{k} (z(x_1) - z(x_1))B_i(x)$$

$$= 0.$$

Now suppose $m > 1$ and $x \in I_m = (x_{m-1}, x_m) = (t_{(m-1)(k-\nu)+k}, t_{m(k-\nu)+1+\nu})$. Then $B_i(x) \neq 0$ iff $i_L := (m-1)(k-\nu) + 1 \leq i \leq m(k-\nu) + \nu =: i_R$. Hence

$$\zeta(x) - z_h(x) = \sum_{i=i_L}^{i_R} (\zeta(x) - \zeta(t_i))B_i(x) = \sum_{i=i_L}^{i_R} (z(x_m) - \zeta(t_i))B_i(x) .$$

For each i in the range of summation we have $\zeta(t_i) = z(x_j)$ for some j, $j \geq \max(1, m+1-k) =: j_m$. Hence $z(x_m) - \zeta(t_i) = \int_{x_j}^{x_m} z'(s)ds$ (the integral exists because $x_j \geq x_1 > 0$), so we get

$$|z(x_m) - \zeta(t_i)| \leq (\int_{x_j}^{x_m} s^\alpha z'(s)^2 ds)^{1/2} \cdot (\int_{x_j}^{x_m} s^{-\alpha} ds)^{1/2} .$$

Hence for $x \in I_m$,

$$x^\alpha(\zeta(x) - z_h(x))^2 \leq x^\alpha \sum_{i=i_L}^{i_R} (z(x_m) - \zeta(t_i))^2 B_i(x)$$

$$\leq x^\alpha \sum_{i=i_L}^{i_R} (\int_{x_j}^{x_m} s^\alpha z'(s)^2 ds) \cdot (\int_{x_j}^{x_m} s^{-\alpha} ds) B_i(x)$$

$$\leq \|s^{\alpha/2} z'\|^2_{L_2(x_{j_m}, x_m)} (x_m/x_{j_m})^\alpha (x_m - x_{j_m}) \sum_{i=i_L}^{i_R} B_i(x)$$

$$\leq \|s^{\alpha/2} z'\|^2_{L_2(x_{j_m}, x_m)} M(m/j_m)^{\alpha\gamma} \cdot kh$$

$$\leq \|s^{\alpha/2} z'\|^2_{L_2(x_{j_m}, x_m)} Mk^{\alpha\gamma} \cdot kh .$$

Integrating over I_m yields

$$\|x^{\alpha/2}(\zeta - z_h)\|^2_{L_2(I_m)} \le Mk^{\alpha\gamma}kh^2 \|s^{\alpha/2}z'\|^2_{L_2(x_{j_m}, x_m)} .$$

Summing over m gives

$$\|x^{\alpha/2}(\zeta - z_h)\|^2_{L_2} \le Mk^{\alpha\gamma}k^2h^2 \|x^{\alpha/2}z'\|^2_{L_2} ,$$

which, combined with (3.6), finishes the proof.

Lemma 3.3. Let $f \in L_2(I)$, and define g via $g(x) = x^{-1}\int_0^x f(t)dt$. Then $g \in L_2(I)$, and $\|g\|_{L_2} \le 2\|f\|_{L_2}$.

Proof. See [9, p. 532] or [11, p. 240].

We now show how Nitsche's trick works in our situation. Let u be the solution of Problem (P), u_h the solution of Problem (P_h), and assume $x_j/x_i \le M(j/i)^\gamma$ for $1 \le i \le j$. Then we have

Lemma 3.4. $\|x^{\alpha/2}(u - u_h)\|_{L_2} \le Ch\|u - u_h\|_E$, where $C = C(M,\gamma,\alpha,k)$.

Proof. Let $e(x) := u(x) - u_h(x)$. Let $z(x)$ solve $-(x^\alpha z')' = x^\alpha e$ on I, $z(1) = z'(0) = 0$. Then

$$(z,v)_E = \int_0^1 x^\alpha e(x)v(x)dx \tag{3.7}$$

for all $v \in \overset{\circ}{H}{}^1_\alpha(I)$. For $x \in I$ we have the formula

$$z(x) = \int_x^1 t^{-\alpha}\int_0^t s^\alpha e(s)dsdt . \tag{3.8}$$

Differentiating (3.8) twice yields

$$x^{\alpha/2}z''(x) = \alpha x^{-1}\int_0^x (s/x)^{\alpha/2}s^{\alpha/2}e(s)ds - x^{\alpha/2}e(x)$$

and hence, by Lemma (3.3), $x^{\alpha/2}z'' \in L_2(I)$ and

$$\|x^{\alpha/2}z''\|_{L_2} \le (2\alpha + 1)\|x^{\alpha/2}e\|_{L_2} . \tag{3.9}$$

Now, by (3.3) and (3.7) we have for all $z_h \in \overset{\circ}{S}{}^h$,

$$\text{(3.10)} \qquad \begin{aligned} \|x^{\alpha/2}e\|^2_{L_2} &= (z,e)_E \\ &= (z - z_h, e)_E \\ &\le \|e\|_E \|z - z_h\|_E \\ &= \|e\|_E \|x^{\alpha/2}(z' - z_h')\|_{L_2} . \end{aligned}$$

Note that the space $(\overset{\circ}{S}{}^h)' = \{v_h' : v_h \in \overset{\circ}{S}{}^h\}$ is exactly equal to the space $S^h_{k-1,\nu-1}$. Furthermore, it is easy to see from (3.8) that $z' \in H^1_\alpha(I)$. Thus we may apply Lemma 3.2 to find a function $\zeta_h \in S^h_{k-1,\nu-1}$ (and hence a $z_h \in \overset{\circ}{S}{}^h$ with $z_h' = \zeta_h$) such that

$$\text{(3.11)} \qquad \|x^{\alpha/2}(z' - z_h')\|_{L_2} = \|x^{\alpha/2}(z' - \zeta_h)\|_{L_2} \le C(M,\gamma,\alpha,k)h\| x^{\alpha/2}z''\|_{L_2} .$$

The relations (3.9), (3.10) and (3.11) yield the result.

The argument contained in Lemma 3.4 obviously gives an error bound of the form $\|u - u_h\|_E \le Ch\|x^{\alpha/2}f\|_{L_2}$; thus we have estimates for $\|x^{\alpha/2}(u' - u_h')\|_{L_2}$ and $\|x^{\alpha/2}(u - u_h)\|_{L_2}$. It is more interesting to consider the quantity $\|u - u_h\|_{L_p}$ for $p = 2$ and $p = \infty$, as an a priori bound on these would ensure the error could not be badly behaved at $x = 0$, at least for reasonably smooth u. Our first step is to bound $u' - u_h'$.

<u>Theorem 3.5</u>. Assume the partition (x_i) satisfies $x_j/x_i \le M(j/i)^\gamma$ for $1 \le i \le j$, $M > 0$, $\gamma \ge 1$. Then there exists a $C = C(M,\gamma,\alpha,k) > 0$ such that

$$\text{(3.12)} \qquad \|u' - u_h'\|_{L_2} \le C \inf_{v_h \in \overset{\circ}{S}{}^h} \|u' - v_h'\|_{L_2} ,$$

and

$$\text{(3.13)} \qquad \|u' - u_h'\|_{L_\infty} \le CM_{\underline{\underline{t}}}^{1/2} \inf_{v_h \in \overset{\circ}{S}{}^h} \|u' - v_h'\|_{L_\infty} ,$$

where $M_{\underline{\underline{t}}}$ is the global mesh ratio defined after (2.1).

<u>Proof</u>. Let $P_\alpha : L_2(I) \to S^h_{k-1,\nu-1}$ via $\int_0^1 x^\alpha(w - P_\alpha w)v_h dx = 0$ for all $v_h \in S^h_{k-1,\nu-1}$. Since $(\overset{\circ}{S}{}^h)' = S^h_{k-1,\nu-1}$, (3.3) implies $u_h' = P_\alpha(u')$. Thus

$$u' - u_h' = (I - P_\alpha)u'$$
$$= (I - P_\alpha)(u' - v_h')$$

for all $v_h \in \overset{\circ}{S}{}^h$. Hence for $p = 2,\infty$, Theorem 2.1 gives

$$\|u' - u_h'\|_{L_p} \le \|I - P_\alpha\|_p \inf_{v_h \in \overset{\circ}{S}{}^h} \|u' - v_h'\|_{L_p}$$
$$\le (1 + \|P_\alpha\|_p) \inf_{v_h \in \overset{\circ}{S}{}^h} \|u' - v_h'\|_{L_p}$$
$$\le CM_{\underline{\underline{t}}}^{\frac{1}{2}-\frac{1}{p}} \inf_{v_h \in \overset{\circ}{S}{}^h} \|u' - v_h'\|_{L_p} ,$$

and the proof is complete.

<u>Corollary 3.6</u>. (a) If $u \in H^j(I)$ where $1 \le j \le k$, then

$$\|u' - u_h'\|_{L_2} \le Ch^{j-1} \|u^{(j)}\|_{L_2} .$$

(b) If $u \in W^{j,\infty}(I) := \{v : v, v', \ldots, v^{(j)} \in L_\infty(I)\}$ where $1 \le j \le k$, then

$$\|u' - u_h'\|_{L_\infty} \le CM_{\underline{\underline{t}}}^{1/2} h^{j-1} \|u^{(j)}\|_{L_\infty} .$$

<u>Proof</u>. The Corollary is an immediate consequence of (3.12), (3.13) and well-known approximation properties of the spaces $S^h_{k,\nu}$ (see, e.g., [16]).

We now use an argument like that of Lemma 3.4 to give an a priori bound on $\|u - u_h\|_{L_2}$.

<u>Theorem 3.7</u>. Under the hypotheses of Theorem 3.5, there exists a constant $C = C(M,\gamma,\alpha,k) > 0$ such that

(3.14) $$\|u - u_h\|_{L_2} \le Ch \|u' - u_h'\|_{L_2} .$$

<u>Proof</u>. Let $e(x) := u(x) - u_h(x)$, and let $z(x)$ solve $-(x^\alpha z')' = e$ on $(0,1)$, $z(1) = 0$. The boundary condition to impose at 0 is $\lim_{x \to 0+} x^\alpha z'(x) = 0$. Then there is a unique solution z, given by

(3.15) $$z(x) = \int_x^1 t^{-\alpha} \int_0^t e(s)\,ds .$$

Differentiating twice and using Lemma 3.3, we find $x^\alpha z'' \in L_2(I)$ and $\|x^\alpha z''\|_{L_2} \leq (2\alpha + 1)\|e\|_{L_2}$. Now,

$$\|e\|^2_{L_2} = -\int (x^\alpha z')'e = \int x^\alpha z'e' = \int x^\alpha (z' - z_h')e'$$

for all $z_h \in \overset{\circ}{S}{}^h$, using integration by parts and (3.3). By Lemma 3.2, there is a $z_h \in \overset{\circ}{S}{}^h$ with $\|x^\alpha(z' - z_h')\|_{L_2} \leq Ch\|x^\alpha z''\|_{L_2}$, $C = C(M,\gamma,\alpha,k)$. Thus

$$\|e\|^2_{L_2} \leq \|x^\alpha(z' - z_h')\|_{L_2}\|e'\|_{L_2} \leq Ch\|x^\alpha z''\|_{L_2}\|e'\|_{L_2}$$
$$\leq Ch(2\alpha + 1)\|e\|_{L_2}\|e'\|_{L_2} .$$

<u>Corollary 3.8</u>. If $u \in H^j(I)$ where $1 \leq j \leq k$, then $\|u - u_h\|_{L_2} \leq Ch^j\|u^{(j)}\|_{L_2}$, so we have an optimal convergence rate in L_2.

The Corollary is an immediate consequence of (3.14) and Lemma 3.6.

We now turn to the more delicate matter of obtaining an estimate for $\|u - u_h\|_{L_\infty}$. To begin, let $z \in (0,1)$ and let $G(x) = G_z(x)$ be the Green's function for (3.1), so

$$(3.16)\quad G(x) = \begin{cases} z^{1-\alpha} - 1, & 0 \leq x < z \\ x^{1-\alpha} - 1, & z \leq x \leq 1 \end{cases} \quad (\alpha \neq 1), \qquad G(x) = \begin{cases} \log 1/z, & 0 \leq x < z \\ \log 1/x, & z \leq x \leq 1 \end{cases} \quad (\alpha = 1) .$$

Then for all $G_h \in \overset{\circ}{S}{}^h$,

$$e(z) = \int_0^1 x^\alpha G'(x)e'(x)dx$$
$$= \int_0^1 x^\alpha (G'(x) - G_h'(x))e'(x)dx ,$$

so

$$(3.17)\qquad |e(z)| \leq \|e'\|_{L_\infty} \inf_{G_h \in \overset{\circ}{S}{}^h} \int_0^1 x^\alpha |G'(x) - G_h'(x)|dx .$$

We have Theorem 3.5 to bound the first factor here, so our goal is to bound the second factor. Our method will be to construct a suitable "interpolant" of G. The next theorem gives the result. First let us define, for a given partition $0 = x_0 < x_1 < \dots < x_N = 1$ and $k + 1 \le r \le N$,

$$S_r := \sum_{j=r}^{N} x_{j-k}^{1-k} \max(x_j - x_{j-k}, x_{j+k-1} - x_{j-1})^k , \tag{3.18}$$

where $x_j := x_N = 1$ for $j > N$. Note that if $x_j = (j/N)^\gamma$, then

$$\gamma - k < -1 \Rightarrow S_r \le C(k,\gamma)h^\gamma (r - k)^{\gamma-k+1}$$

$$\gamma - k = -1 \Rightarrow S_r \le C(k,\gamma)h^\gamma |\log(N/(r - k))|$$

$$\gamma - k > -1 \Rightarrow S_r \le C(k,\gamma)h^{k-1} .$$

<u>Theorem 3.9</u>. Let $x_{J-1} < z \le x_J$ $(1 \le J \le N)$ and let $G(x) = G_z(x)$ be the Green's function given in (3.16). Assume the mesh satisfies $x_j/x_i \le M(j/i)^\gamma$, $1 \le i \le j$. Then there exists a constant $C = C(M,k,\alpha,\gamma) > 0$ such that

$$\inf_{G_h \in \overset{\circ}{S}{}^h} \int_0^1 x^\alpha |G'(x) - G_h'(x)| dx \le C(M,k,\alpha,\gamma)(h + S_{J+k}) . \tag{3.19}$$

In particular, if the mesh is quasi-uniform (i.e., $x_j - x_{j-1} \ge \mu h$ for $1 \le j \le N$, so $\gamma = 1$), then

$$\inf_{G_h \in \overset{\circ}{S}{}^h} \int_0^1 x^\alpha |G'(x) - G_h'(x)| dx \le C(\mu,k,\alpha) \cdot \begin{cases} h \log(1/x_J), & k = 2 \\ h , & k \ge 3 . \end{cases} \tag{3.20}$$

<u>Corollary 3.10</u>. If $u \in W^{j,\infty}(I)$ where $1 \le j \le k$, then $\|e\|_{L_\infty} \le C(M,k,\alpha,\gamma)M_{\underline{\underline{t}}}^{1/2}(h^j + h^{j-1}S_{k+1})\|u^{(j)}\|_{L_\infty}$. In particular, if the mesh is quasi-uniform then

$$\|e\|_{L_\infty} \le C(M,k,\alpha)\|u^{(j)}\|_{L_\infty} \cdot \begin{cases} h^j|\log h|, & k = 2 \\ h^j , & k \ge 3 \end{cases} .$$

Corollary 3.10 follows immediately from Corollary 3.6, (3.17), and Theorem 3.9. Its content is that we have an optimal rate of convergence in L_∞ except for the case $k = 2$ of piecewise linear trial functions S^h. This is the same result as found in [15]. An example will be presented later to show the $|\log h|$ factor cannot be removed if $k = 2$. First we prove Theorem 3.9.

Proof of Theorem 3.9. The proof is somewhat lengthy, but straightforward. We construct a map $T : L_1 \to S^h$ such that $Tf(x)$ depends only on the values of f in a neighborhood of x, i.e., T is local. Then we let $G_h = TG$. Now, G consists of two smooth pieces joined together continuously at z. Hence the problem of estimating $G'(x) - G_h'(x)$ can be divided into three cases, depending on whether x is sufficiently far to the left of z, sufficiently far to the right of z, or in the vicinity of z. The estimates for these cases are combined to yield the result.

To begin, let $\{\lambda_i\}_1^n \subset L_1'$ with $\lambda_i g = \int_{t_i}^{t_{i+k}} f_i g\, dx$, $\lambda_i B_j = \delta_{ij}$ and $\|f_i\|_{L_\infty} \le D_k(t_{i+k} - t_i)^{-1}$; such functions f_i are constructed in [4] and are used in Lemma 2.5 above.

For $f \in L_1$, define $Tf = \sum_{i=1}^{n} (\lambda_i f) B_i$. Then T maps L_1 to S^h. We would put $G_h = TG$ except we require $G_h \in \overset{\circ}{S}{}^h$. Thus, define $\bar{G}_h = TG$, $G_h = \bar{G}_h - (\lambda_n G) B_n$. Because $B_i(1) = 0$ for $1 \le i < n$, we have $G_h \in \overset{\circ}{S}{}^h$. Now,

$$\|x^\alpha(G' - G_h')\|_{L_1} \le \|x^\alpha(G' - \bar{G}_h')\|_{L_1} + \|x^\alpha(\bar{G}_h' - G_h')\|_{L_1} . \tag{3.21}$$

Let us note for future reference the inequality

$$\int |B_i'(x)|\, dx \le 2 , \tag{3.22}$$

which follows from the equality (see (4.6) of [5])

$$B_{i,k}' = \frac{k-1}{t_{i+k-1} - t_i} B_{i,k-1} - \frac{k-1}{t_{i+k} - t_{i+1}} B_{i+1,k-1}$$

expressing the derivative of the i^{th} B-spline of order k in terms of B-splines of order $k - 1$.

We can easily estimate the second integral on the right hand side of (3.21). For, $\bar{G}_h - G_h = (\lambda_n G) B_n$ and

$$|\lambda_n G| = \left| \int_{t_n}^{t_{n+k}} f_n(x) G(x)\, dx \right| \le D_k G(t_n) .$$

But $t_n = x_{N-1} \ge 1 - h$ and thus

$$G(t_n) \le \begin{Bmatrix} t_n^{1-\alpha} - 1, & \alpha \ne 1 \\ \\ \log(1/t_n), & \alpha = 1 \end{Bmatrix} \le C(\alpha)h \ .$$

Thus $|\lambda_n G| \le C(\alpha,k)h$, and so

$$\|x^\alpha(\bar{G}_h' - G_h')\|_{L_1} = \int x^\alpha |\lambda_n G| |B_n'(x)| dx \le 2C(\alpha,k)h \ . \tag{3.23}$$

Now comes the task of estimating the other term on the right hand side of (3.21). For fixed x and f sufficiently smooth, we have $f(s) = P(s) + R_x(s)$, where $P(s) := \sum_{r=0}^{k-1} (s - x)^r f^{(r)}(x)/r!$. It is clear from the definition $Tf = \Sigma(\lambda_i f)B_i$ and the fact that $\lambda_i B_j = \delta_{ij}$ that T preserves polynomials of order k, hence

$$(Tf - f)'(x) = (TR_x - R_x)'(x) = (TR_x)'(x) = \sum_i (\lambda_i R_x) B_i'(x) \ ,$$

so

$$(\bar{G}_h - G)'(x) = \sum_i (\lambda_i R_x) B_i'(x)$$

where $R_x(s)$ is G minus its Taylor polynomial expansion at the point x (which exists for $x \ne z$).

Suppose $x \in I_m$ with $m \le J - k$. (Recall $z \in I_J$.) Then, with $i_R := (m - 1)(k - \nu) + k$ and $i_L := (m - 1)(k - \nu) + 1$,

$$(\bar{G}_h - G)'(x) = \sum_{i=i_L}^{i_R} (\lambda_i R_x) B_i'(x) \ , \tag{3.24}$$

since $\operatorname{supp} B_i \cap I_m \ne \phi$ if and only if $i_L \le i \le i_R$. For $i \le i_R$, $\lambda_i R_x = \int_{t_i}^{t_{i+k}} f_i(s) R_x(s) ds = 0$ since $R_x(s) = 0$ for $x < z$ and $s < z$. Thus $(\bar{G}_h - G)' = 0$ on I_m for $m \le J - k$, which is the result for x sufficiently far to the left of z.

Now suppose $x \in I_m$ with $m \ge J + k$. Again we have (3.24). Furthermore, for $i_L \le i \le i_R$,

$$(3.25)\qquad \sup_{t_i \le s \le t_{i+k}} |R_x(s)| = \sup_{t_i \le s \le t_{i+k}} \left|\int_s^x (x-\xi)^{k-1} G^{(k)}(\xi)\,d\xi/(k-1)!\right|$$

$$\le C(\alpha,k)\, t_{i_L}^{1-\alpha-k} \sup_{t_i \le s \le t_{i+k}} \left|\int_s^x (x-\xi)^{k-1} d\xi\right|$$

$$\le C(\alpha,k)\, t_{i_L}^{1-\alpha-k} \max(x_m - t_{i_L}, t_{i_R+k} - x_{m-1})^k$$

$$\le C(\alpha,k)\, x_{m-k}^{1-\alpha-k} \max(x_m - x_{m-k}, x_{m+k-1} - x_{m-1})^k ,$$

since $G^{(k)}(\xi) = c_{k,\alpha}\xi^{1-\alpha-k}$ and $t_{i_L} \ge x_{m-k}$, $t_{i_R+k} \le x_{m+k-1}$.

Hence for $i_L \le i \le i_R$,

$$|\lambda_i R_x| = \left|\int_{t_i}^{t_{i+k}} f_i(s) R_x(s)\,ds\right|$$

$$\le D_k \cdot C(\alpha,k)\, x_{m-k}^{1-\alpha-k} \max(x_m - x_{m-k}, x_{m+k-1} - x_{m-1})^k ,$$

and so

$$(3.26)\qquad \int_{I_m} x^\alpha |G' - \bar{G}_h'|\,dx \le \sum_{i_L}^{i_R} |\lambda_i R_x| \int_{I_m} x^\alpha |B_i'(x)|\,dx$$

$$\le C(\alpha,k)\, x_m^\alpha x_{m-k}^{1-\alpha-k} \max(x_m - x_{m-k}, x_{m+k-1} - x_{m-1})^k$$

$$\le C(M,k,\alpha,\gamma)\, x_{m-k}^{1-k} \max(x_m - x_{m-k}, x_{m+k-1} - x_{m-1})^k ,$$

where we used the mesh hypothesis $x_m/x_{m-k} \le M(m/(m-k))^\gamma$.

Summing (3.26) for $J + k \le m \le N$ gives

$$(3.27)\qquad \int_{x_{J+k-1}}^1 x^\alpha |G' - \bar{G}_h'|\,dx \le C(M,k,\alpha,\gamma) S_{J+k} ,$$

which is the result for x sufficiently far to the right of z.

Finally we must consider the intermediate case $x \in I_m$, $J - k < m < J + k$. Write

$$\bar{G}_h(x) - G(x) = \sum_i (\lambda_i G) B_i(x) - G(z) + G(z) - G(x)$$

$$= \sum_i \lambda_i (G - G(z)) B_i(x) + G(z) - G(x) ,$$

and thus

$$(3.28)\qquad \int_{I_m} x^\alpha |\bar{G}_h' - G'|dx \le \sum_i |\lambda_i(G - G(z))| \int_{I_m} x^\alpha |B_i'(x)|dx + \int_{I_m} x^\alpha |G'(x)|dx$$

$$\le \sum_{i_L}^{i_R} |\lambda_i(G - G(z))| x_m^\alpha \cdot 2 + c_\alpha h ,$$

where i_L, i_R are defined as before.

For $i_L \le i \le i_R$,

$$|\lambda_i(G - G(z))| = \left|\int_{t_i}^{t_{i+k}} f_i(s)(G(s) - G(z))ds\right|$$

$$= \left|\int_{z}^{t_{i+k}} f_i(s)(G(s) - G(z))ds\right|$$

$$\le D_k(t_{i+k} - t_i)^{-1} c_\alpha z^{-\alpha} |t_{i+k} - z|^2$$

$$\le C(k,\alpha)hz^{-\alpha} ,$$

and thus

$$(3.29)\qquad 2\sum_{i_L}^{i_R} |\lambda_i(G - G(z))| x_m^\alpha \le C(k,\alpha)h(x_m/z)^\alpha .$$

If $z \ge x_1$ (i.e., $J \ge 2$), we have

$$(x_m/z)^\alpha \le (x_{J+k-1}/x_{J-1})^\alpha \le C(M,\alpha,\gamma)$$

and hence (3.29) is bounded by $C(M,k,\alpha,\gamma)h$. This coupled with (3.28) proves (3.19) in the case $z \ge x_1$.

Finally, suppose $0 < z < x_1$. In this case define

$$\tilde{G}(x) = \begin{cases} G(x), & x_1 \le x \le 1 \\ G(x_1), & 0 \le x < x_1 \end{cases} .$$

By all that has gone before,

$$(3.30)\qquad \inf_{G_h \in \mathring{S}^h} \int_0^1 x^\alpha |\tilde{G}' - G_h'|dx \le C(M,k,\alpha,\gamma)(h + S_{k+1}) .$$

Also,

$$(3.31) \qquad \int_0^1 x^\alpha |G' - \hat{G}'| dx \le \int_0^{x_1} x^\alpha c_\alpha x^{-\alpha} dx \le c_\alpha h .$$

Combining (3.30) and (3.31) proves (3.19) in the case $0 < z < x_1$, and thus Theorem 3.9 is proven. It may be worthwhile to note that letting $z \to 0+$ proves that (3.19) holds even for $z = 0$.

Example. Here is an example that shows the $|\log h|$ factor cannot be removed from the upper bound of Corollary 3.10 if $k = 2$. Let $x_j = j/N$, $0 \le j \le N$, $h = N^{-1}$, $k = 2$, $u(x) = 1 - x^2$. Let $u_j := u_h(x_j)$, $0 \le j \le N$. Then $u_h(x)$ is given by (3.3),

$$\int_0^1 x^\alpha (u - u_h)' v_h' dx = 0 \quad \text{for all } v_h \in \overset{\circ}{S}{}^h ,$$

and $u_N = 0$. For v_h we may choose a function such that $v_h' = 1$ on I_j, $v_h' = 0$ off I_j. (The fact there is only one boundary condition to impose on v_h makes this possible.) Doing so we find

$$\int_{I_j} x^\alpha \left(\frac{u_j - u_{j-1}}{h} \right) dx = \int_{I_j} x^\alpha (-2x) dx ,$$

and hence

$$\frac{u_j - u_{j-1}}{h} = - \frac{2(\alpha + 1)}{\alpha + 2} h \frac{j^{\alpha+2} - (j - 1)^{\alpha+2}}{j^{\alpha+1} - (j - 1)^{\alpha+1}} ,$$

$$u_{j-1} = u_j + 2 \frac{\alpha + 1}{\alpha + 2} h^2 \frac{j^{\alpha+2} - (j - 1)^{\alpha+2}}{j^{\alpha+1} - (j - 1)^{\alpha+1}} .$$

Since $u_N = 0$, we see

$$u_m = 2 \frac{\alpha + 1}{\alpha + 2} h^2 \sum_{j=m+1}^{N} \frac{j^{\alpha+2} - (j - 1)^{\alpha+2}}{j^{\alpha+1} - (j - 1)^{\alpha+1}} .$$

In particular,

$$u_0 = 2\,\frac{\alpha + 1}{\alpha + 2}\,h^2 \sum_{j=1}^{N} \left[\frac{\alpha + 2}{\alpha + 1}\,(j - \tfrac{1}{2}) + \frac{\alpha(\alpha + 2)}{12(\alpha + 1)}\,(j - \tfrac{1}{2})^{-1} + O(j^{-2})\right]$$

$$= 2h^2 \sum_{j=1}^{N} (j - \tfrac{1}{2}) + \frac{\alpha h^2}{6} \sum_{j=1}^{N} (j - \tfrac{1}{2})^{-1} + O(h^2)$$

$$= 1 + \frac{\alpha h^2}{6}\,|\log h| + O(h^2)$$

$$= u(0) + \frac{\alpha h^2}{6}\,|\log h| + O(h^2)\ ,$$

so $\|u_h - u\|_{L_\infty} \geq ch^2|\log h|$, and Corollary 3.10 is sharp.

§4. Extensions and Numerical Results

The purpose of this section is to extend the results of the previous section to nonlinear equations. Our results will not be as complete or satisfying as those of §2 and §3; in particular, we will assume $1 \le \alpha \le 2$ and will not obtain L_∞ error estimates. Finally, we will show some numerical results.

We begin with a lemma which will be useful later.

Lemma 4.1. Suppose $x_1 \ge c_1 h$ as $h \to 0$ where $c_1 > 0$ is independent of h. Then there exists a constant $C = C(k,c_1)$ such that for any $\alpha \ge 1$ and any $\varphi \in S^h = S^h_{k,\nu}$,

$$\int_0^1 x^\alpha \varphi(x)^2 dx \ge C(k,c_1) h^\alpha \int_0^1 \varphi(x)^2 dx \ . \tag{4.1}$$

Proof. For any $\varphi \in S^h$, $\int_0^{x_1} x^\alpha \varphi(x)^2 dx \ge c_k x_1^\alpha \int_0^{x_1} \varphi(x)^2 dx$ since φ is a polynomial of order k on $[0,x_1]$ and all norms are equivalent on a finite-dimensional space. Thus

$$\int_0^1 x^\alpha \varphi^2 = \int_{x_1}^1 x^\alpha \varphi^2 + \int_0^{x_1} x^\alpha \varphi^2$$

$$\ge x_1^\alpha \int_{x_1}^1 \varphi^2 + c_k x_1^\alpha \int_0^{x_1} \varphi^2$$

$$\ge c_1^\alpha \min(1,c_k) h^\alpha \int_0^1 \varphi^2 \ .$$

As a first step to the nonlinear problem we investigate the effect of lower order terms on the linear problem. Consider the equation

$$-(x^\alpha u')' + x^\alpha q(x) u = x^\alpha f(x) \quad \text{on} \quad (0,1) \tag{4.2}$$

$$u(1) = 0, \quad u'(0) = 0 \ .$$

The equation (4.2) comes from $-u'' - \frac{\alpha}{x} u' + qu = f$ after multiplication by x^α. We seek an approximation $U_h \in \overset{\circ}{S}{}^h$ of u as the solution of

$$\int_0^1 x^\alpha U_h' v_h' dx + \int_0^1 x^\alpha q U_h v_h dx = \int_0^1 x^\alpha f v_h dx \quad \text{for all} \quad v_h \in \overset{\circ}{S}{}^h \ , \tag{4.3}$$

equivalently,

$$(4.4) \qquad \int_0^1 x^\alpha (u' - U_h')v_h' dx + \int_0^1 x^\alpha q(u - U_h)v_h dx = 0 \quad \text{for all} \quad v_h \in \mathring{S}^h .$$

Let us make the following assumptions.

(4.5) There are $c_0 > 0$, $Q > 0$ such that $(c_0 - 1)\lambda_0^2 \le q_0 \le q(x) \le Q$ for all $x \in [0,1]$, where λ_0^2 is the smallest positive eigenvalue of the problem $-(x^\alpha v')' = \lambda^2 x^\alpha v$ on $(0,1)$, $v(1) = 0 = v'(0)$.

(4.6) There are constants $c_1 > 0$, $M > 0$, $\gamma \ge 1$ such that $x_1 \ge c_1 h$ and $x_j/x_i \le M(j/i)^\gamma$ for $1 \le i \le j$.

By (4.5), unique solutions u to (4.2) and U_h to (4.3) exist.

Let $u_h \in \mathring{S}^h$ be given by $\int_0^1 x^\alpha (u' - u_h')v_h' dx = 0$ for all $v_h \in \mathring{S}^h$. Section 3 gives us estimates on $\|u - u_h\|$ in various norms, hence to estimate $\|u - U_h\|$ in those norms it suffices to bound $\|u_h - U_h\|$.

The equations defining u_h and U_h may be combined to yield

$$(4.7) \qquad \int_0^1 x^\alpha (u_h' - U_h')v_h' dx + \int_0^1 x^\alpha q(u_h - U_h)v_h dx = \int_0^1 x^\alpha q(u_h - u)v_h dx$$

for all $v_h \in \mathring{S}^h$.

We also note that for $v \in \mathring{H}_\alpha^1(I)$,

$$(4.8) \qquad \begin{aligned} \int x^\alpha v'^2 + \int x^\alpha q v^2 &\ge \int x^\alpha v'^2 + (c_0 - 1)\lambda_0^2 \int x^\alpha v^2 \\ &\ge \int x^\alpha v'^2 + (\underline{c} - 1)\lambda_0^2 \int x^\alpha v^2 \\ &\ge \int x^\alpha v'^2 + (\underline{c} - 1) \int x^\alpha v'^2 \\ &\ge \underline{c} \int x^\alpha v'^2 , \end{aligned}$$

where $\underline{c} := \min(c_0, 1)$, and also

$$(4.9) \qquad \int x^\alpha v'^2 + \int x^\alpha q v^2 \ge c_0 \lambda_0^2 \int x^\alpha v^2 .$$

We now state the main results for the estimation of $u_h - U_h$.

<u>Theorem 4.1</u>. Under the preceding assumptions, we have

$$(4.10) \qquad \|x^{\alpha/2}(u_h - U_h)\|_{L_2} + \|u_h - U_h\|_E \le C(c_0, \lambda_0, Q) \| x^{\alpha/2}(u - u_h)\|_{L_2} ,$$

(4.11) $$\|u_h' - U_h'\|_{L_2} \le C(c_0,c_1,\lambda_0,Q,k)h^{-\alpha/2}\|u - u_h\|_{L_2} ,$$

(4.12) $$\|u - U_h\|_{L_2} \le C(c_0,c_1,\lambda_0,q,M,\alpha,k,\gamma)h(\|u' - u_h'\|_{L_2} + h^{-\alpha/2}\|u - u_h\|_{L_2}) .$$

Thus if $1 \le \alpha \le 2$ we have optimal convergence rates for $\|u - U_h\|_{L_2}$ and $\|u' - U_h'\|_{L_2}$.

Proof. Let $E_h := u_h - U_h$. Using E_h for v_h in (4.7) gives

(4.13) $$\int x^\alpha E_h'^2 + \int x^\alpha qE_h^2 = \int x^\alpha q(u_h - u)E_h \le \|x^{\alpha/2}E_h\|_{L_2} Q\,\|x^{\alpha/2}(u_h - u)\|_{L_2} .$$

Hence (4.10) follows from (4.8) and (4.9).

For (4.11), we have by Lemma 4.1 and (4.10),

$$C(k,c_1)h^\alpha \int E_h'^2 \le \int x^\alpha E_h'^2 = \|E_h\|_E^2 \le C(c_0,\lambda_0,Q)^2\|u - u_h\|_{L_2}^2 .$$

For (4.12) we use a Nitsche-type argument. Let z be the (unique) solution of $-(x^\alpha z')' + x^\alpha qz = u - U_h$ on $(0,1)$ with $z(1) = 0$, $\lim_{x\to 0+} x^\alpha z(x) = 0$. Then $\|x^\alpha z''\|_{L_2} \le C(q,\alpha)\|u - U_h\|_{L_2}$. By (4.4),

$$\|u - U_h\|_{L_2}^2 = \int x^\alpha z'(u' - U_h') + \int x^\alpha qz(u - U_h)$$

$$= \int x^\alpha(z' - z_h')(u' - U_h') + \int x^\alpha q(z - z_h)(u - U_h)$$

for all $z_h \in \overset{\circ}{S}{}^h$. By Lemma 3.2 we may choose $z_h' \in S^h_{k-1,\nu-1}$ (which is equivalent to choosing $z_h \in \overset{\circ}{S}{}^h$) such that $\|x^\alpha(z' - z_h')\|_{L_2} \le C(M,\alpha,k,\gamma)h\,\|x^\alpha z''\|_{L_2}$. We also have $\|x^\alpha(z - z_h)\|_{L_2} \le \Lambda^{-1}\|x^\alpha(z' - z_h')\|_{L_2}$ where $\Lambda = \Lambda(\alpha)$ is the smallest eigenvalue of $-(x^{2\alpha}v')' = \lambda x^{2\alpha}v$ on $(0,1)$, $v(1) = 0 = v'(0)$. Hence

$$\|u - U_h\|_{L_2}^2 \le \|u' - U_h'\|_{L_2} Ch\,\|x^\alpha z''\|_{L_2} + Q\Lambda^{-1}Ch\,\|x^\alpha z''\|_{L_2}\|u - U_h\|_{L_2}$$

$$\le C(q,M,\alpha,k,\gamma)h\,\|u - U_h\|_{L_2}(\|u' - U_h'\|_{L_2} + \|u - U_h\|_{L_2}) ,$$

and thus for sufficiently small h

$$\|u - U_h\|_{L_2} \le \tilde{C}(q,M,\alpha,k,\gamma)h \|u' - U_h'\|_{L_2}$$

$$\le \tilde{C}h(\|u' - u_h'\|_{L_2} + \|u_h' - U_h'\|_{L_2})$$

$$\le Ch(\|u' - u_h'\|_{L_2} + h^{-\alpha/2} \|u - u_h\|_{L_2}) ,$$

where the last step used (4.11). This completes the proof.

We now turn to the nonlinear problem

(4.14) $$-(x^{\alpha}u')' = x^{\alpha}f(x,u) \quad \text{on} \quad (0,1), \qquad u(1) = 0 = u'(0) .$$

We will assume $f(x,t)$ is smooth as a function of x and differentiable as a function of t, with partial derivative satisfying $|\frac{\partial f}{\partial t}(x,t)| \le K$, and for some $c_0 > 0$, $\frac{\partial f}{\partial t}(x,t) \le (1 - c_0)\lambda_0^2$, where λ_0^2 was defined in (4.5). Then one may show (as in [1] or [12] for the unweighted problem) there exists a unique solution $u \in \mathring{H}^1_\alpha$ to (4.14), and u satisfies

(4.15) $$\int x^{\alpha}u'v'dx = \int x^{\alpha}f(x,u(x))v(x)dx$$

for all $v \in \mathring{H}^1_\alpha$. Similarly it can be shown there is a unique $u_h \in \mathring{S}^h$ which satisfies

(4.16) $$\int x^{\alpha}u_h'v_h'dx = \int x^{\alpha}f(x,u_h(x))v_h(x)dx$$

for all $v_h \in \mathring{S}^h$. Our goal is to estimate $u - u_h$.

To do so we consider an auxiliary problem defined as follows. Let $q(x) := -\frac{\partial f}{\partial u}(x,u(x))$ and let $w \in \mathring{H}^1_\alpha$ solve

(4.17) $$-(x^{\alpha}w')' + x^{\alpha}qw = x^{\alpha}(qu + f(x,u(x))), \quad w(1) = w'(0) = 0 .$$

By the assumption on f we have $q \ge (c_0 - 1)\lambda_0^2$ and hence (4.17) has a unique solution, which is clearly u. Now consider the problem: find $\bar{u}_h \in \mathring{S}^h$ such that for all $v_h \in \mathring{S}^h$,

(4.18) $$\int x^{\alpha}\bar{u}_h'v_h' + \int x^{\alpha}q\bar{u}_hv_h = \int x^{\alpha}[quv_h + f(x,u)v_h] .$$

For simplicity, let us assume $1 \le \alpha \le 2$ and $u \in H^k(I)$. Then by Theorem 4.1,

(4.19) $$\|u - \bar{u}_h\|_{L_2} + h\|u' - \bar{u}_h'\|_{L_2} \le Ch^k \|u^{(k)}\|_{L_2} ,$$

where C is independent of h. Notice that this implies

(4.20) $$\|u - \bar{u}_h\|_{L_\infty} \le Ch^{k-1} \|u^{(k)}\|_{L_2} .$$

Let us finally assume $f(x,t)$ is twice differentiable with respect to t, and $|\partial^2 f/\partial t^2| \le K$. This plus the previous assumptions yield the main result.

Theorem 4.2. Let u solve (4.14) and u_h solve (4.16). Then $\|u - u_h\|_{L_2} + h\|u' - u_h'\|_{L_2} \le Ch^k \|u^{(k)}\|_{L_2}$ where C is independent of h.

Proof. For all $v_h \in \overset{\circ}{S}{}^h$, we have for some $0 < \theta, \bar{\theta} < 1$,

(4.21) $$\begin{aligned} \int x^\alpha \bar{u}_h' v_h' - \int x^\alpha f(x,\bar{u}_h)v_h &= \int x^\alpha [qu - q\bar{u}_h + f(x,u) - f(x,\bar{u}_h)]v_h \\ &= \int x^\alpha (u - \bar{u}_h)[-\frac{\partial f}{\partial u}(x,u) + \frac{\partial f}{\partial u}(x,u + \theta(\bar{u}_h - u))]v_h \\ &= -\int x^\alpha (u - \bar{u}_h)^2 \frac{\partial^2 f}{\partial u^2}(x,u + \bar{\theta}(\bar{u}_h - u))v_h \\ &\le K\int x^\alpha (u - \bar{u}_h)^2 |v_h| . \end{aligned}$$

We also have, for all $v_h \in \overset{\circ}{S}{}^h$,

(4.22) $$\begin{aligned} &\int x^\alpha \bar{u}_h' v_h' - \int x^\alpha f(x,\bar{u}_h)v_h \\ &= \int x^\alpha \bar{u}_h' v_h' - \int x^\alpha u_h' v_h' - \int x^\alpha f(x,\bar{u}_h)v_h + \int x^\alpha f(x,u_h)v_h \\ &= \int x^\alpha (\bar{u}_h' - u_h')v_h' - \int x^\alpha (\bar{u}_h - u_h)v_h \frac{\partial f}{\partial u}(x,\bar{u}_h + \theta(u_h - \bar{u}_h)) . \end{aligned}$$

Putting $v_h = \bar{u}_h - u_h$ in (4.22) yields

$$\begin{aligned} &\int x^\alpha \bar{u}_h'(\bar{u}_h - u_h)' - \int x^\alpha f(x,\bar{u}_h)(\bar{u}_h - u_h) \\ &= \int x^\alpha (\bar{u}_h' - u_h')^2 - \int x^\alpha (\bar{u}_h - u_h)^2 \frac{\partial f}{\partial u}(x,\bar{u}_h + \theta(u_h - \bar{u}_h)) \\ &\ge \int x^\alpha (\bar{u}_h' - u_h')^2 + (c_0 - 1)\lambda_0^2 \int x^\alpha (\bar{u}_h - u_h)^2 \\ &\ge \underline{c} \int x^\alpha (\bar{u}_h' - u_h')^2 , \end{aligned}$$

where $\underline{c} := \min(c_0, 1)$, as in (4.8). Thus, by (4.21)

$$\underline{c} \int x^{\alpha}(\bar{u}_h' - u_h')^2 \leq \int x^{\alpha}\bar{u}_h'(\bar{u}_h - u_h)' - \int x^{\alpha} f(x,\bar{u}_h)(\bar{u}_h - u_h)$$

$$\leq K \int x^{\alpha}(u - \bar{u}_h)^2 |\bar{u}_h - u_h|$$

$$\leq K \|u - \bar{u}_h\|_{L_\infty} \|x^{\alpha/2}(u - \bar{u}_h)\|_{L_2} \|x^{\alpha/2}(\bar{u}_h - u_h)\|_{L_2}$$

$$\leq K\lambda_0^{-1} \|u - \bar{u}_h\|_{L_\infty} \|x^{\alpha/2}(u - \bar{u}_h)\|_{L_2} \|x^{\alpha/2}(\bar{u}_h' - u_h')\|_{L_2} ,$$

and thus

$$\|x^{\alpha/2}(\bar{u}_h' - u_h')\|_{L_2} \leq K\lambda_0^{-1}\underline{c}^{-1} \|u - \bar{u}_h\|_{L_\infty} \|x^{\alpha/2}(u - \bar{u}_h)\|_{L_2}$$

and also

$$\|x^{\alpha/2}(\bar{u}_h - u_h)\|_{L_2} \leq K\lambda_0^{-2}\underline{c}^{-1} \|u - \bar{u}_h\|_{L_\infty} \|x^{\alpha/2}(u - \bar{u}_h)\|_{L_2} .$$

Recalling Lemma 4.1, we obtain with the help of (4.19) and (4.20),

$$\|\bar{u}_h - u_h\|_{L_2} + \|\bar{u}_h' - u_h'\|_{L_2} \leq C(k,c_1)h^{-\alpha/2}(\|x^{\alpha/2}(\bar{u}_h - u_h)\|_{L_2} + \|x^{\alpha/2}(\bar{u}_h' - u_h')\|_{L_2})$$

$$\leq Ch^{-\alpha/2} \|u - \bar{u}_h\|_{L_\infty} \|x^{\alpha/2}(u - \bar{u}_h)\|_{L_2}$$

$$\leq Ch^{-\alpha/2}h^{2k-1} \|u^{(k)}\|_{L_2}^2 .$$

Thus,

$$\|u - u_h\|_{L_2} + h\|u' - u_h'\|_{L_2} \leq \|u - \bar{u}_h\|_{L_2} + h\|u' - \bar{u}_h'\|_{L_2} + \|\bar{u}_h - u_h\|_{L_2} + \|\bar{u}_h' - u_h\|_{L_2}$$

$$\leq Ch^k \|u^{(k)}\|_{L_2} + Ch^{2k-1-\alpha/2} \|u^{(k)}\|_{L_2}^2$$

$$\leq Ch^k \|u^{(k)}\|_{L_2}(1 + h^{k-1-\alpha/2} \|u^{(k)}\|_{L_2}) ,$$

and the proof is complete, since $k \geq 2$ and $1 \leq \alpha \leq 2$.

Finally, here are some numerical results. The Ritz-Galerkin procedure was programmed and tested on the three nonlinear problems which follow. The mesh was taken to be uniform with mesh width h, and S^h was taken to be C^2 cubic splines ($k = 4$, $\nu = 3$). de Boor's package for calculating with B-splines [3] was used to handle the spline manipulations. The nonlinear problem (4.16) was tackled with Newton's method and an initial guess of 0 for the solution. The integrals involved in the linear problems to which Newton's method leads were evaluated with a composite Gaussian quadrature rule using 5 points per interval. The iterations were judged to have converged when the residual was about at the unit roundoff level (which is roughly 10^{-18} in double precision on the Univac 1110). The errors in the L_2 and L_∞ norms were then estimated by an evaluation at 40 equally spaced points per interval. In the tables we present the error (the notation .1734(-5) means $.1734 \times 10^{-5}$), the rate, which is defined by $\log(e(h_1)/e(h_2))/\log(h_1/h_2))$, and an estimate of the constant C in the inequality $\|e\|_{L_p} \leq Ch^k \|u^{(iv)}\|_{L_p}$, defined by $C_p = \|e\|_{L_p} h^{-k} / \|u^{(iv)}\|_{L_p}$, $p = 2$ and $p = \infty$. For each example $\|u^{(iv)}\|_{L_\infty}$ was determined analytically and $\|u^{(iv)}\|_{L_2}$ was determined by numerical integration, specifically, 48-point Gaussian quadrature.

Example 1. $\alpha = 1$, $f(x,u) = -\frac{64}{49} e^u$.

The solution is $u(x) = 2 \log 7 - 2 \log(8 - x^2)$. We have $\|u^{(iv)}\|_{L_2} \doteq .6198$, $\|u^{(iv)}\|_{L_\infty} \doteq 1.130$. Each case converged in 4 iterations.

h	$\|e\|_{L_2}$	rate	C_2	$\|e\|_{L_\infty}$	rate	C_∞
1/4	.1734(-5)		.716(-3)	.3388(-5)		.768(-3)
1/6	.3745(-6)	3.78	.783(-3)	.7806(-6)	3.62	.895(-3)
1/8	.1235(-6)	3.86	.816(-3)	.2805(-6)	3.56	.102(-2)
1/10	.5180(-7)	3.89	.836(-3)	.1237(-6)	3.67	.109(-2)
1/12	.2536(-7)	3.92	.848(-3)	.6266(-7)	3.73	.115(-2)
1/16	.8174(-8)	3.94	.864(-3)	.2106(-7)	3.79	.122(-2)
1/20	.3384(-8)	3.95	.874(-3)	.8945(-8)	3.84	.127(-2)

Example 2. $\alpha = 1$, $f(x,u) = e^u$.

This is a well-known example (see, e.g. [13]). A solution (not unique) is $u(x) = 2 \log(1 + B) - 2 \log(1 + Bx^2)$, where $B := 3 - 2\sqrt{2}$. The theoretical results do not include this example ($\frac{\partial f}{\partial u}$ has the wrong sign), but no convergence difficulty was encountered. Each case required 4 iterations. We have $\|u^{(iv)}\|_{L_2} \doteq .5685$, $\|u^{(iv)}\|_{L_\infty} \doteq .7065$.

h	$\|e\|_{L_2}$	rate	C_2	$\|e\|_{L_\infty}$	rate	C_∞
1/4	.1565(-5)		.705(-3)	.3530(-5)		.128(-2)
1/6	.3154(-6)	3.95	.719(-3)	.7465(-6)	3.83	.137(-2)
1/8	.1007(-6)	3.97	.726(-3)	.2403(-6)	3.94	.139(-2)
1/10	.4150(-7)	3.97	.730(-3)	.9904(-7)	3.97	.140(-2)
1/12	.2010(-7)	3.98	.733(-3)	.4791(-7)	3.98	.141(-2)

<u>Example 3</u>. $\alpha = 2$, $f(x,u) = 3\sqrt{3} + (u + \frac{1}{2}\sqrt{3}x^2)^5$.

This is a modification of an example in [13]. The solution is $u(x) = (1 + x^2/3)^{-1/2} - \sqrt{3}\,x^2/2$. Once again, the theory does not cover this example, but no numerical difficulty was encountered. Each case took 6 iterations. We have $\|u^{(iv)}\|_{L_2} \doteq .8481$, $\|u^{(iv)}\|_{L_\infty} = 1$.

h	$\|e\|_{L_2}$	rate	C_2	$\|e\|_{L_\infty}$	rate	C_∞
1/4	.2107(-5)		.636(-3)	.5237(-5)		.134(-2)
1/6	.4076(-6)	4.05	.623(-3)	.1075(-5)	3.91	.139(-2)
1/8	.1269(-6)	4.06	.613(-3)	.3416(-6)	3.99	.140(-2)
1/10	.5153(-7)	4.04	.608(-3)	.1400(-6)	4.00	.140(-2)
1/12	.2473(-7)	4.03	.605(-3)	.6750(-7)	4.00	.140(-2)

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This paper is concerned with the application of the Ritz-Galerkin method to the numerical solution of singular boundary value problems of the type arising when Poisson's equation on a domain with cylindrical or spherical symmetry is reduced to a one-dimensional problem. The objective is to derive a priori L_2 and L_∞-norm estimates for the error. The difficulty is that these norms are not natural norms for the reduced problem. With the aid of B-splines we prove some nonstandard approximation - theoretic results and use these to derive the desired error estimates. Some numerical results are presented.

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